Generalized Inverses of Morphisms with Kernels

Donald W. Robinson
Department of Mathematics
Brigham Young University
Provo, Utah 84602

and

Roland Puystjens
Seminarie voor Algebra en Functionaalanalyse
Rijksuniversiteit Gent
Galglaan 2, 9000 Gent, België

Submitted by Hans Schneider

ABSTRACT

Let $\phi: X \to Y$ be a morphism with kernel $\kappa: K \to X$ in an additive category with an involution $*$. Then ϕ has a Moore-Penrose inverse $\phi^\dagger$ with respect to $*$ iff $\phi\phi^* + \kappa^*\kappa$ is invertible; in this case, $\phi^\dagger = \phi^* (\phi\phi^* + \kappa^*\kappa)^{-1}$. If $X = Y$, then ϕ has a group inverse $\phi^\#$ iff ϕ has a cokernel $\gamma: X \to K$ and $\phi^2 + \gamma\kappa$ is invertible; in this case, $\phi^\# = \phi\ (\phi^2 + \gamma\kappa)^{-1}$.

INTRODUCTION

Let $\phi: X \to Y$ be a morphism with kernel $\kappa: K \to X$ in an additive category $\mathscr{C}$ with an involution $*$. If ϕ is invertible, then $\kappa = 0$, $\phi\phi^*$ is invertible, and $\phi^{-1} = \phi^* (\phi\phi^*)^{-1}$. On the other hand, if $\phi\phi^*$ is invertible, then $\kappa = 0$, ϕ has a Moore-Penrose inverse $\phi^\dagger$, and $\phi^\dagger = \phi^* (\phi\phi^*)^{-1}$. We extend these two facts by showing that ϕ has a Moore-Penrose inverse $\phi^\dagger$ with respect to $*$ if and only if $\phi\phi^* + \kappa^*\kappa$ is invertible in $\mathscr{C}$; in this case, $\phi^\dagger = \phi^* (\phi\phi^* + \kappa^*\kappa)^{-1}$.

Next, let $\phi: X \to X$ be a morphism in an additive category. Suppose that i is a nonnegative integer and that $\kappa: K \to X$ is a kernel of $\phi^i: X \to X$. Then ϕ

is invertible if and only if ϕ^{i+1} is invertible; in this case, $\phi^{-1} = \phi^i\ (\phi^{i+1})^{-1}$, $\kappa = 0$, and $0: X \to K$ is a cokernel of ϕ^i. We extend these facts and show that ϕ has a Drazin inverse ϕ^D and $i \geqslant$ Drazin index of ϕ if and only if ϕ^i has a cokernel $\gamma: X \to K$ and $\phi^{i+1} + \gamma\kappa$ is invertible; in this case, $\phi^D = \phi^i\ (\phi^{i+1} + \gamma\kappa)^{-1}$. In particular, ϕ with kernel $\kappa: K \to X$ has a group inverse $\phi^{\#}$ if and only if ϕ has a cokernel $\gamma: X \to K$ and $\phi^2 + \gamma\kappa$ is invertible; in this case, $\phi^{\#} = \phi\ (\phi^2 + \gamma\kappa)^{-1}$.

We provide two applications of these theorems. First, we establish some results on bordered inverses. In particular, we show that a morphism $\phi: X \to Y$ with kernel $\kappa: K \to X$ and a cokernel $\lambda: Y \to L$ in an additive category $\mathscr{C}$ with involution * has a Moore-Penrose inverse with respect to * if and only if

$$\begin{pmatrix} \phi & \kappa^* \\ \lambda^* & 0 \end{pmatrix}$$

is invertible in the category of matrices over $\mathscr{C}$. This extends an observation of J. W. Blattner [3]. An analogous theorem for the Drazin inverse is also given. We conclude with some remarks about EP morphisms.

The reader is referred to [25], [26], and [28] for the preliminaries and background of this paper.

1. MOORE-PENROSE INVERSES

Let $\phi: X \to Y$ be a morphism of a category $\mathscr{C}$ with an involution *. (See, for example, [26, p. 131].) Then ϕ is said to have a Moore-Penrose inverse with respect to * provided that there is a morphism $\phi^{\dagger}: Y \to X$ of $\mathscr{C}$ such that

$$\phi\phi^{\dagger}\phi = \phi, \qquad \phi^{\dagger}\phi\phi^{\dagger} = \phi^{\dagger}, \qquad (\phi\phi^{\dagger})^* = \phi\phi^{\dagger}, \qquad (\phi^{\dagger}\phi)^* = \phi^{\dagger}\phi.$$

If such a $\phi^{\dagger}$ exists, then it is unique and is called the Moore-Penrose inverse of ϕ with respect to *. (See, for example, [26, p. 132]; for earlier references see [19] and [24].)

THEOREM 1. *Let $\phi: X \to Y$ be a morphism in an additive category with an involution *. If $\kappa: K \to X$ is a kernel of ϕ, then ϕ has a Moore-Penrose inverse $\phi^{\dagger}$ with respect to * if and only if*

$$\phi\phi^* + \kappa^*\kappa: X \to X$$

is invertible. In this case, κ also has a Moore-Penrose inverse $\kappa^{\dagger}$, $\kappa\kappa^*$ is *invertible,*

$$\kappa^{\dagger}=\kappa^{*}(\kappa\kappa^{*})^{-1}=(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}\kappa^{*},$$

and

$$\phi^{\dagger}=\phi^{*}(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}.$$

*Dually, if $\lambda: Y \to L$ is a cokernel of ϕ, then ϕ has a Moore-Penrose inverse $\phi^{\dagger}$ with respect to * if and only if*

$$\phi^{*}\phi+\lambda\lambda^{*}: Y \to Y$$

is invertible. In this case, λ also has a Moore-Penrose inverse $\lambda^{\dagger}$, $\lambda^{*}\lambda$ is invertible,

$$\lambda^{\dagger}=(\lambda^{*}\lambda)^{-1}\lambda^{*}=\lambda^{*}(\phi^{*}\phi+\lambda\lambda^{*})^{-1},$$

and

$$\phi^{\dagger}=(\phi^{*}\phi+\lambda\lambda^{*})^{-1}\phi^{*}.$$

Proof. Suppose that $\phi\phi^{*}+\kappa^{*}\kappa$ is invertible. Since $\kappa\phi=0$ and $\phi^{*}\kappa^{*}=0$, then

$$(\phi\phi^{*}+\kappa^{*}\kappa)\phi\phi^{*}=\phi\phi^{*}\phi\phi^{*}=\phi\phi^{*}(\phi\phi^{*}+\kappa^{*}\kappa),$$

and

$$\phi\phi^{*}(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}=(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}\phi\phi^{*}. \tag{1}$$

Also, since $\phi\phi^{*}\phi=(\phi\phi^{*}+\kappa^{*}\kappa)\phi$, then

$$(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}\phi\phi^{*}\phi=\phi. \tag{2}$$

We now show that $\phi^{*}(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}$ satisfies the Moore-Penrose equations. Indeed, since $\phi\phi^{*}+\kappa^{*}\kappa$ is symmetric with respect to *, so is $(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}$, and, therefore, so also is

$$\left[\phi^{*}(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}\right]\phi=\phi^{*}(\phi\phi^{*}+\kappa^{*}\kappa)^{-1}\phi.$$

Next, by (1),

$$\begin{aligned}\left\{\phi\left[\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\right]\right\}^* &= \left[\phi\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\right]^*\\ &= \left[(\phi\phi^*+\kappa^*\kappa)^{-1}\right]^*(\phi\phi^*)^* = (\phi\phi^*+\kappa^*\kappa)^{-1}\phi\phi^*\\ &= \phi\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1} = \phi\left[\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\right].\end{aligned}$$

By this result and (2), it follows that

$$\begin{aligned}\phi\left[\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\right]\phi &= \left[\phi\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\right]^*\phi\\ &= \left[(\phi\phi^*+\kappa^*\kappa)^{-1}\right]^*(\phi\phi^*)^*\phi\\ &= (\phi\phi^*+\kappa^*\kappa)^{-1}\phi\phi^*\phi = \phi.\end{aligned}$$

Finally, by (1) and (2),

$$\begin{aligned}&\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\cdot\phi\cdot\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\\ &\quad= \phi^*\left[(\phi\phi^*+\kappa^*\kappa)^{-1}\phi\phi^*\right](\phi\phi^*+\kappa^*\kappa)^{-1}\\ &\quad= \phi^*\left[\phi\phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}\right](\phi\phi^*+\kappa^*\kappa)^{-1}\\ &\quad= \left[(\phi\phi^*+\kappa^*\kappa)^{-1}\phi\phi^*\phi\right]^*(\phi\phi^*+\kappa^*\kappa)^{-1}\\ &\quad= \phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}.\end{aligned}$$

Consequently, ϕ has the Moore-Penrose inverse

$$\phi^\dagger = \phi^*(\phi\phi^*+\kappa^*\kappa)^{-1}$$

with respect to the involution $*$.

Conversely, suppose that $\phi^\dagger$ exists with respect to $*$. Since $(1_X - \phi\phi^\dagger)\phi = 0$ and κ is a kernel of ϕ, then $1_X - \phi\phi^\dagger = \xi\kappa$ for some $\xi: X \to K$. Since $\kappa\phi = 0$, then $\kappa\xi\kappa = \kappa(1_X - \phi\phi^\dagger) = \kappa = 1_K\kappa$; since κ is monic, then $\kappa\xi = 1_K$. Consequently, since $\kappa\xi = 1_K$ and $\xi\kappa = 1_X - \phi\phi^\dagger$ is symmetric with respect to $*$, then ξ is the Moore-Penrose inverse of κ with respect to $*$. That is, $\kappa^\dagger$ exists, $\kappa^\dagger = \xi$, $\kappa\kappa^\dagger = 1_K$, and $\phi\phi^\dagger + \kappa^\dagger\kappa = 1_X$.

In particular, $\kappa^{*\dagger}$ also exists, and

$$(\kappa\kappa^*)(\kappa^{*\dagger}\kappa^\dagger) = \kappa(\kappa^*\kappa^{*\dagger})\kappa^\dagger = \kappa(\kappa^\dagger\kappa)^*\kappa^\dagger$$

$$= \kappa(\kappa^\dagger\kappa)\kappa^\dagger = (\kappa\kappa^\dagger)(\kappa\kappa^\dagger) = 1_K 1_K = 1_K.$$

Since $\kappa\kappa^*$ is symmetric with respect to $*$, it follows that $\kappa\kappa^*$ is invertible with inverse $\kappa^{*\dagger}\kappa^\dagger$; also

$$\kappa^\dagger = \kappa^\dagger\kappa\kappa^\dagger = (\kappa^\dagger\kappa)^*\kappa^\dagger = (\kappa^*\kappa^{*\dagger})\kappa^\dagger$$

$$= \kappa^*(\kappa^{*\dagger}\kappa^\dagger) = \kappa^*(\kappa\kappa^*)^{-1}.$$

Finally, since $\kappa\phi = 0$, then

$$\phi\phi^*\phi^{*\dagger}\phi^\dagger = \phi(\phi^\dagger\phi)^*\phi^\dagger = \phi\phi^\dagger\phi\phi^\dagger = \phi\phi^\dagger,$$

$$\kappa^*\kappa\phi^{*\dagger}\phi^\dagger = \kappa^*\kappa\left[\phi(\phi^*\phi)^\dagger\right]\phi^\dagger = \kappa^*(\kappa\phi)(\phi^*\phi)^\dagger\phi^\dagger = 0,$$

$$\phi\phi^*\kappa^\dagger\kappa^{*\dagger} = \phi\phi^*\left[\kappa^*(\kappa\kappa^*)^{-1}\right]\kappa^{*\dagger} = \phi(\kappa\phi)^*(\kappa\kappa^*)^{-1}\kappa^{*\dagger} = 0,$$

$$\kappa^*\kappa\kappa^\dagger\kappa^{*\dagger} = \kappa^*(\kappa\kappa^\dagger)^*\kappa^{*\dagger} = (\kappa\kappa^\dagger\kappa)^*\kappa^{*\dagger} = (\kappa^\dagger\kappa)^* = \kappa^\dagger\kappa,$$

and it follows that

$$(\phi\phi^* + \kappa^*\kappa)(\phi^{*\dagger}\phi^\dagger + \kappa^\dagger\kappa^{*\dagger}) = \phi\phi^\dagger + \kappa^\dagger\kappa = 1_X.$$

Again, since $\phi\phi^* + \kappa^*\kappa$ is symmetric with respect to $*$, then $\phi\phi^* + \kappa^*\kappa$ is invertible with inverse $\phi^{*\dagger}\phi^\dagger + \kappa^\dagger\kappa^{*\dagger}$. Moreover,

$$(\phi\phi^* + \kappa^*\kappa)^{-1}\kappa^* = (\phi^{*\dagger}\phi^\dagger + \kappa^\dagger\kappa^{*\dagger})\kappa^*$$

$$= \phi^{*\dagger}\phi^\dagger\kappa^* + \kappa^\dagger\kappa^{*\dagger}\kappa^* = \phi^{*\dagger}(\phi^*\phi)^\dagger\phi^*\kappa^* + \kappa^\dagger(\kappa\kappa^\dagger)^*$$

$$= \phi^{*\dagger}(\phi^*\phi)^\dagger(\kappa\phi)^* + \kappa^\dagger\kappa\kappa^\dagger = 0 + \kappa^\dagger = \kappa^\dagger.$$

A dual argument completes the proof of the theorem. ∎

COROLLARY 1.1. *Let the conditions be as in Theorem* 1. *If* $\kappa: K \to X$ *is a kernel and* $\lambda: Y \to L$ *is a cokernel of* $\phi: X \to Y$, *then* $\phi\phi^* + \kappa^*\kappa: X \to X$ *is invertible if and only if* $\phi^*\phi + \lambda\lambda^*: Y \to Y$ *is invertible.*

Proof. The conclusion is clear, since by Theorem 1 each statement is equivalent to the existence of $\phi^\dagger$. ∎

EXAMPLE 1.1. Let $\phi = (2,0): 1 \to 2$ be a morphism in the category $\mathscr{M}_{\mathbf{Z}}$ of finite matrices over the integers $\mathbf{Z}$ with the involution of transpose. Then

$$\kappa = 0: 0 \to 1, \qquad \lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix}: 2 \to 1$$

are a kernel and a cokernel, respectively, of ϕ; but neither

$$\phi\phi^* + \kappa^*\kappa = (4) + (0) = (4): 1 \to 1$$

nor

$$\phi^*\phi + \lambda\lambda^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}(2 \quad 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}: 2 \to 2$$

is invertible in $\mathscr{M}_{\mathbf{Z}}$. Indeed, ϕ does not have a Moore-Penrose inverse in $\mathscr{M}_{\mathbf{Z}}$. We note, however, that both

$$\kappa\kappa^* = 0: 0 \to 0, \qquad \lambda^*\lambda = (1): 1 \to 1$$

are invertible. In particular, this shows that the invertibility of $\kappa\kappa^*$ and/or $\lambda^*\lambda$ in the statement of Theorem 1 is a necessary but not sufficient condition for the existence of $\phi^\dagger$.

EXAMPLE 1.2. Let $\mathscr{C}$ be the category consisting of the single object ∞, morphisms which are infinite matrices over $\mathbf{Z}$ with at most a finite number of nonzero entries in each row and each column, composition of matrix multiplication and addition, and involution of transpose. Let $\phi = (\phi_{ij}): \infty \to \infty$ have every entry zero except $\phi_{2p,p} = 1$, $p = 1, 2, \ldots$. Then $\kappa = (k_{ij}): \infty \to \infty$ with every entry zero except $k_{p,2p-1} = 1$, $p = 1, 2, \ldots$, is a kernel of ϕ. In particular, $\phi\phi^* + \kappa^*\kappa = 1_\infty$ is invertible in $\mathscr{C}$, and $\phi^\dagger = \phi^*$. However, ϕ does not have a cokernel in $\mathscr{C}$. Indeed, if $\lambda: \infty \to \infty$ is such that $\phi\lambda = 0$, then $\lambda = 0: \infty \to \infty$, which is not epic.

The ϕ of Example 1.2 is itself an epic morphism. Thus, $(\phi, \infty, 1_\infty)$ is an (epic, monic) factorization of ϕ. Since $\phi^*\phi = 1_\infty$ and $1_\infty 1_\infty^* = 1_\infty$, then by

Theorem 3 of [26, p. 135], the Moore-Penrose inverse of ϕ is given by $1_\infty^*(1_\infty 1_\infty^*)^{-1}(\phi^*\phi)^{-1}\phi^* = \phi^*$. This example also serves as an illustration of the following result.

COROLLARY 1.2. *Let the conditions be as in Theorem* 1. *If* $\kappa: K \to X$ *is a kernel and* (ϕ_1, Z, ϕ_2) *is an (epic, monic) factorization of* $\phi: X \to Y$, *then* $\phi\phi^* + \kappa^*\kappa: X \to X$ *is invertible iff* $\phi_1^*\phi_1: Z \to Z$ *and* $\phi_2\phi_2^*: Z \to Z$ *are invertible.*

Proof. By Theorem 1 above and Theorem 3 of [26, p. 135], each statement is equivalent to the existence of $\phi^\dagger$. ■

EXAMPLE 1.3. Let $\mathscr{C}$ consist of a single object X and two morphisms 1 and 0 with modulo 2 arithmetic. Clearly $1^\dagger = 1$ and $0^\dagger = 0$. Now, the morphism 1 has an (epic, monic) factorization $(1, X, 1)$ but has no kernel nor cokernel in $\mathscr{C}$. On the other hand, the morphism 0 has kernel 1 and cokernel 1 but no (epic, monic) factorization in $\mathscr{C}$.

EXAMPLE 1.4. Let $\mathscr{C}$ be the subcategory of $\mathscr{M}_{\mathbf{Z}}$ consisting of the following objects and morphisms:

$$(a) \circlearrowright 1 \underset{\binom{a}{0}}{\overset{(a,0)}{\rightleftarrows}} 2 \circlearrowleft \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in \mathbf{Z}$. Then

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \quad 0)$$

has an (epic, monic) factorization, but no kernel nor cokernel in $\mathscr{C}$; clearly, $\phi^\dagger = \phi$. On the other hand,

$$\phi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

has kernel $\kappa = (1,0): 1 \to 2$ and cokernel

$$\lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix}: 2 \to 1,$$

but does not possess an (epic, monic) factorization in $\mathscr{C}$; again, $\phi^\dagger = \phi$.

2. DRAZIN INVERSES

Let $\phi: X \to X$ be a morphism of a category $\mathscr{C}$. Then ϕ is said to have a Drazin inverse provided that there is a morphism $\phi^D: X \to X$ and a nonnegative integer i such that

$$\phi^i \phi^D \phi = \phi^i, \qquad \phi^D \phi \phi^D = \phi^D, \qquad \phi \phi^D = \phi^D \phi.$$

The least such i is called the Drazin index of ϕ. (See [6].) If such a ϕ^D and i exist, then ϕ^D and the Drazin index are unique.

A group inverse is a Drazin inverse of index at most 1. In this case, the Drazin inverse is denoted by $\phi^\#$ and satisfies

$$\phi \phi^\# \phi = \phi, \qquad \phi^\# \phi \phi^\# = \phi^\#, \qquad \phi \phi^\# = \phi^\# \phi.$$

(See also [2, p. 162], [7], [8, p. 120], [18, p. 273], [28], and [29, p. 158].)

THEOREM 2. *Let $\phi: X \to X$ be a morphism of an additive category $\mathscr{C}$, and let $i \geqslant 0$ be an integer. If $\kappa: K \to X$ is a kernel of $\phi^i: X \to X$, then ϕ has a Drazin inverse ϕ^D in $\mathscr{C}$ and $i \geqslant$ Drazin index of ϕ if and only if ϕ^i has a cokernel $\lambda: X \to L$, $\kappa\lambda: K \to L$ is invertible, and $\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa: X \to X$ is invertible. In this case, $\gamma = \lambda(\kappa\lambda)^{-1}: X \to K$ is a cokernel of ϕ^i, $\phi\phi^D + \gamma\kappa = 1_X$, and*

$$\phi^D = \phi^i \left(\phi^{i+1} + \gamma\kappa\right)^{-1} = \left(\phi^{i+1} + \gamma\kappa\right)^{-1} \phi^i.$$

Proof. Let $\lambda: X \to L$ be a cokernel of ϕ^i with $\kappa\lambda: K \to L$ and $\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa: X \to X$ both invertible. For $s \geqslant 0$ an integer,

$$\left[\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa\right]\phi^{i+s} = \phi^{2i+s+1} = \phi^{i+s}\left[\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa\right].$$

Consequently,

$$\phi^{i+s} = \left[\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{2i+s+1} = \phi^{2i+s+1}\left[\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}$$

and

$$\begin{aligned}\phi^{i+s}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1} &= \left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{2i+s+1}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\\ &= \left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{i+s}.\end{aligned}$$

From these facts we deduce that ϕ has a Drazin inverse by demonstrating that

$$\phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}=\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{i}$$

satisfies the Drazin equations $\phi^i x\phi=\phi^i$, $x\phi x=x$, $\phi x=x\phi$. Specifically,

$$\begin{aligned}\phi^{i}\left\{\phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\right\}\phi &= \phi^{i+i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi\\ &= \left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{2i+1}=\phi^{i},\end{aligned}$$

$$\begin{aligned}&\left\{\phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\right\}\phi\left\{\phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\right\}\\ &\quad= \left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{2i+1}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\\ &\quad= \phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1},\end{aligned}$$

$$\begin{aligned}\phi\left\{\phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\right\} &= \phi^{i+1}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\\ &= \left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{i+1}\\ &= \left\{\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^{i}\right\}\phi\\ &= \left\{\phi^{i}\left[\phi^{i+1}+\lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\right\}\phi.\end{aligned}$$

That is, ϕ has the Drazin inverse

$$\phi^D = \phi^i\left[\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1} = \left[\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi^i$$

and $i \geqslant$ Drazin index.

Conversely, suppose that ϕ^D exists and that $i \geqslant$ Drazin index of ϕ. Since $\phi^i(1_X - \phi^D\phi) = \phi^i - \phi^i\phi^D\phi = 0$, then $1_X - \phi^D\phi = \gamma\kappa$ for some $\gamma: X \to K$. We show that such a γ is a cokernel of ϕ^i. Indeed, since $(\phi^i\gamma)\kappa = \phi^i(\gamma\kappa) = \phi^i(1_X - \phi^D\phi) = \phi^i - \phi^i\phi^D\phi = 0$ and κ is monic, then $\phi^i\gamma = 0$. If $\phi^i\eta = 0$ for some $\eta: X \to N$, then $\eta = [1_X - (\phi^D)^i\phi^i]\eta = [1_X - (\phi^D\phi)^i]\eta = (1_X - \phi^D\phi)\eta = (\gamma\kappa)\eta = \gamma(\kappa\eta)$. Finally, since $(\kappa\gamma)\kappa = \kappa(\gamma\kappa) = \kappa(1_X - \phi^D\phi) = \kappa(1_X - \phi\phi^D) = \kappa[1_X - (\phi\phi^D)^i] = \kappa[1_X - \phi^i(\phi^D)^i] = \kappa 1_X = \kappa = 1_K\kappa$ and κ is monic, then $\kappa\gamma = 1_K$; in particular, γ is epic.

Consequently, $\gamma: X \to K$ is a cokernel of ϕ^i with $\kappa\gamma = 1_K$ invertible. Also,

$$\left(\phi^{i+1} + \gamma\kappa\right)\left[\left(\phi^D\right)^{i+1} + \gamma\kappa\right] = \left(\phi\phi^D\right)^{i+1} + \gamma\kappa\gamma\kappa = \phi\phi^D + \gamma\kappa = 1_X.$$

Similarly, $[(\phi^D)^{i+1} + \gamma\kappa](\phi^{i+1} + \gamma\kappa) = 1_X$, and $\phi^{i+1} + \gamma\kappa$ is invertible with inverse $(\phi^D)^{i+1} + \gamma\kappa$.

Moreover,

$$\phi^i\left(\phi^{i+1} + \gamma\kappa\right)^{-1} = \phi^i\left[\left(\phi^D\right)^{i+1} + \gamma\kappa\right] = \left(\phi\phi^D\right)^i\phi^D$$

$$= \phi^D = \phi^D\left(\phi^D\phi\right)^i = \left[\left(\phi^D\right)^{i+1} + \gamma\kappa\right]\phi^i = \left(\phi^{i+1} + \gamma\kappa\right)^{-1}\phi^i.$$

■

COROLLARY 2. *Let $\phi: X \to X$ be a morphism of an additive category $\mathscr{C}$. If $\kappa: K \to X$ is a kernel of ϕ, then ϕ has a group inverse in $\mathscr{C}$ if and only if ϕ has a cokernel $\lambda: X \to L$ and both $\kappa\lambda: K \to L$ and $\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa: X \to X$ are invertible. In this case, $\gamma = \lambda(\kappa\lambda)^{-1}: X \to K$ is a cokernel of ϕ, $\phi\phi^\# + \gamma\kappa = 1_X$, and*

$$\phi^\# = \phi\left(\phi^2 + \gamma\kappa\right)^{-1} = \left(\phi^2 + \gamma\kappa\right)^{-1}\phi.$$

Proof. This is the special case of $i = 1$ in Theorem 2. ■

EXAMPLE 2. Let $\mathscr{M}_{\mathbf{Z}/5\mathbf{Z}}$ be the category finite matrices over the integers modulo 5. If

$$\phi = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} : 2 \to 2,$$

then

$$\kappa = \begin{pmatrix} 1 & 4 \end{pmatrix} : 1 \to 2, \qquad \lambda = \begin{pmatrix} 1 \\ 2 \end{pmatrix} : 2 \to 1$$

are a kernel and a cokernel, respectively, of ϕ. Since $\kappa\lambda = (4) : 1 \to 1$ is invertible and

$$\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(4)(1 \quad 4) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} : 2 \to 2$$

is also invertible with inverse

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} : 2 \to 2,$$

then

$$\phi^{\#} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} : 2 \to 2.$$

If

$$\phi = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} : 2 \to 2$$

then

$$\kappa = \begin{pmatrix} 3 & 1 \end{pmatrix} : 1 \to 2, \qquad \lambda = \begin{pmatrix} 3 \\ 1 \end{pmatrix} : 2 \to 1$$

are a kernel and cokernel, respectively, of ϕ. Since $\kappa\lambda = (0) : 1 \to 1$ is not invertible, then ϕ does not have a group inverse. However, since $\phi^2 = 0 : 2 \to 2$ has $\kappa = \lambda = 1 : 2 \to 2$ as a kernel and cokernel, and $\kappa\lambda = 1 : 2 \to 2$ and $\phi^3 + \lambda(\kappa\lambda)^{-1}\kappa = 1 : 2 \to 2$ are invertible, then ϕ has the Drazin inverse $\phi^D = \phi^2[\phi^3 + \lambda(\kappa\lambda)^{-1}\kappa]^{-1} = 0 : 2 \to 2$ with Drazin index 2.

3. BORDERED INVERSES

In 1962 J. W. Blattner [3] proved that if A is an m-by-n complex matrix of rank r, if K is an $(m-r)$-by-n matrix whose rows form a basis of the left null space of A, and if L is an n-by-$(n$-$r)$ matrix whose columns form a basis of the right null space of A, then the matrix

$$\begin{pmatrix} A & K^* \\ L^* & 0 \end{pmatrix}$$

is invertible with inverse

$$\begin{pmatrix} A^{\dagger} & L^{*\dagger} \\ K^{*\dagger} & 0 \end{pmatrix}.$$

(See also [10], [27, p. 421], and [2, p. 228]; for some recent applications of this result, see [1], [31], and [9, p. 227]; and for some related studies see [11], [13], and [14].)

In this section we extend the preceding fact, together with some related results, to morphisms in any additive category $\mathscr{C}$ with an involution *. Specifically, generalized invertibility in the sense of Moore-Penrose and Drazin is characterized in terms of ordinary invertibility in the category of finite matrices over $\mathscr{C}$.

Given an additive category $\mathscr{C}$, the category $\mathscr{M}_{\mathscr{C}}$ of finite matrices over $\mathscr{C}$ is defined as follows: the objects are finite lists $(X_1,\ldots,X_m)$ of objects of $\mathscr{C}$, the morphisms are finite matrices

$$(\phi_{ij}):(X_1,\ldots,X_m)\to(Y_1,\ldots,Y_n)$$

with $\phi_{ij}: X_i \to Y_j$, and composition is the usual matrix multiplication and addition. Moreover, if $\mathscr{C}$ is a category with an involution *, then $\mathscr{C}$ is also a category with involution under the usual matrix involution: if (ϕ_{ij}) is as above, then

$$(\phi_{ij})^{\circledast}=(\psi_{ij}):(Y_1,\ldots,Y_n)\to(X_1,\ldots,X_m)$$

where $\psi_{ij}=\phi_{ji}^*$.

We say that a morphism $\phi: X \to Y$ is *-left invertible if there is a morphism $\psi: Y \to X$ such that $\psi\phi = 1_Y$ and $(\phi\psi)^* = \phi\psi$. Similarly, $\phi: X \to Y$ is *-right invertible if there is a $\psi: Y \to X$ such that $\phi\psi = 1_X$, $(\psi\phi)^* = \psi\phi$.

LEMMA. *If $\phi: X \to Y$ is a morphism in a category with an involution* $*$, *then*

(1) ϕ *is* $*$*-left invertible iff* $\phi^*\phi$ *is invertible, and in this case,* $\phi^\dagger = (\phi^*\phi)^{-1}\phi^*$;

(2) ϕ *is* $*$*-right invertible iff* $\phi\phi^*$ *is invertible, and in this case,* $\phi^\dagger = \phi^*(\phi\phi^*)^{-1}$;

(3) ϕ *is invertible iff both* $\phi^*\phi$ *and* $\phi\phi^*$ *are invertible, and in this case,* $\phi^{-1} = (\phi^*\phi)^{-1}\phi^* = \phi^*(\phi\phi^*)^{-1}$.

Proof. (1): Let $\phi^*\phi$ be invertible. Then ϕ has the Moore-Penrose inverse $\phi^\dagger = (\phi^*\phi)^{-1}\phi^*$. In particular, $\phi^\dagger\phi = 1_Y$ and $(\phi\phi^\dagger)^* = \phi\phi^\dagger$. That is, ϕ is $*$-left invertible.

Conversely, suppose $\phi^\dagger: Y \to X$ is such that $\phi^\dagger\phi = 1_Y$ with $(\phi\phi^\dagger)^* = \phi\phi^\dagger$. Then $\phi^\dagger$ is the Moore-Penrose inverse of ϕ and

$$(\phi^\dagger\phi^{*\dagger})(\phi^*\phi) = \phi^\dagger(\phi\phi^\dagger)^*\phi = \phi^\dagger\phi\phi^\dagger\phi = \phi^\dagger\phi = 1_Y,$$

$$(\phi^*\phi)(\phi^\dagger\phi^{*\dagger}) = \phi^*(\phi\phi^\dagger)^*\phi^{\dagger *} = (\phi^\dagger\phi\phi^\dagger\phi)^* = 1_Y^* = 1_Y.$$

That is $\phi^*\phi$ is invertible with inverse $\phi^\dagger\phi^{*\dagger}$.

In this case, ϕ has the Moore-Penrose inverse $\phi^\dagger = (\phi^*\phi)^{-1}\phi^*$.

(2): This result follows by a similar argument.

(3): If ϕ is invertible, then ϕ^* is invertible and so also are $\phi^*\phi$ and $\phi\phi^*$. Conversely, if $\phi^*\phi$ and $\phi\phi^*$ are invertible, then ϕ has a left inverse $(\phi^*\phi)^{-1}\phi^*$ and a right inverse $\phi^*(\phi\phi^*)^{-1}$, and hence invertible with inverse $\phi^{-1} = (\phi^*\phi)^{-1}\phi^* = \phi^*(\phi\phi^*)^{-1}$. ■

THEOREM 3.1. *Let $\phi: X \to Y$ be a morphism of an additive category $\mathscr{C}$ with an involution* $*$. *If $\kappa: K \to X$ is a kernel of ϕ, then ϕ has a Moore-Penrose inverse $\phi^\dagger$ with respect to $*$ if and only if*

$$(\phi, \kappa^*): (X) \to (Y, K)$$

is $^{\circledast}$*-right invertible in $\mathscr{M}_{\mathscr{C}}$. In this case,*

$$(\phi, \kappa^*)^\dagger = \begin{pmatrix} \phi^\dagger \\ \kappa^{*\dagger} \end{pmatrix} = \begin{pmatrix} \phi^*(\phi\phi^* + \kappa^*\kappa)^{-1} \\ \kappa(\phi\phi^* + \kappa^*\kappa)^{-1} \end{pmatrix} : (Y, K) \to (X).$$

If $\lambda : Y \to L$ is a cokernel of ϕ, then ϕ has a Moore-Penrose inverse $\phi^\dagger$ with respect to $*$ if and only if

$$\begin{pmatrix} \phi \\ \lambda^* \end{pmatrix} : (X, L) \to (Y)$$

is $^{\circledast}$-left invertible in $\mathcal{M}_{\mathscr{C}}$. In this case

$$\begin{pmatrix} \phi \\ \lambda^* \end{pmatrix}^\dagger = (\phi^\dagger, \lambda^{*\dagger}) = \big((\phi^*\phi + \lambda\lambda^*)^{-1}\phi^*, (\phi^*\phi + \lambda\lambda^*)^{-1}\lambda\big) : (Y) \to (X, L).$$

Proof. By Theorem 1 and the previous Lemma, $\phi^\dagger$ exists in $\mathscr{C}$ iff $\phi\phi^* + \kappa^*\kappa$ is invertible in $\mathscr{C}$ iff $(\phi, \kappa^*)(\phi, \kappa^*)^{\circledast}$ is invertible in $\mathcal{M}_{\mathscr{C}}$ iff (ϕ, κ^*) is $^{\circledast}$-right invertible in $\mathcal{M}_{\mathscr{C}}$. In this case,

$$\begin{aligned}(\phi, \kappa^*)^\dagger &= (\phi, \kappa^*)^{\circledast}\left[(\phi, \kappa^*)(\phi, \kappa^*)^{\circledast}\right]^{-1} \\ &= \begin{pmatrix} \phi^* \\ \kappa \end{pmatrix}(\phi\phi^* + \kappa^*\kappa)^{-1} = \begin{pmatrix} \phi^*(\phi\phi^* + \kappa^*\kappa)^{-1} \\ \kappa(\phi\phi^* + \kappa^*\kappa)^{-1} \end{pmatrix}\end{aligned}$$

is the Moore-Penrose inverse of (ϕ, κ^*) with respect to $^{\circledast}$.

A similar argument provides a proof of the second part of the theorem. ∎

THEOREM 3.2. *Let $\phi : X \to Y$ be a morphism of an additive category $\mathscr{C}$ with an involution $*$. If $\kappa : K \to X$ is a kernel and $\lambda : Y \to L$ is a cokernel of ϕ, then ϕ has a Moore-Penrose inverse with respect to $*$ if and only if*

$$\Phi = \begin{pmatrix} \phi & \kappa^* \\ \lambda^* & 0 \end{pmatrix} : (X, L) \to (Y, K)$$

is invertible in $\mathcal{M}_{\mathscr{C}}$. In this case,

$$\Phi^{-1} = \begin{pmatrix} \phi^\dagger & \lambda^{*\dagger} \\ \kappa^{*\dagger} & 0 \end{pmatrix} : (Y, K) \to (X, L).$$

Proof. Since $\kappa\phi = 0$, $\phi\lambda = 0$, $\phi^*\kappa^* = 0$, and $\lambda^*\phi^* = 0$, then

$$\Phi\Phi^{\circledast} = \begin{pmatrix} \phi\phi^* + \kappa^*\kappa & 0 \\ 0 & \lambda^*\lambda \end{pmatrix} : (X, L) \to (X, L),$$

$$\Phi^{\circledast}\Phi = \begin{pmatrix} \phi^*\phi + \lambda\lambda^* & 0 \\ 0 & \kappa\kappa^* \end{pmatrix} : (Y, K) \to (Y, K).$$

Consequently, by Theorem 1 and the previous Lemma, $\phi^\dagger$ exists with respect to $*$ iff $\phi\phi^* + \kappa^*\kappa, \lambda^*\lambda, \phi^*\phi + \lambda\lambda^*, \kappa\kappa^*$ are invertible iff $\Phi\Phi^*$ and $\Phi^*\Phi$ are invertible iff Φ is invertible. In this case,

$$\Phi^{-1} = \Phi^{\circledast}(\Phi\Phi^{\circledast})^{-1} = \begin{pmatrix} \phi^* & \lambda \\ \kappa & 0 \end{pmatrix} \begin{pmatrix} (\phi\phi^* + \kappa^*\kappa)^{-1} & 0 \\ 0 & (\lambda^*\lambda)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \phi^\dagger & \lambda^{*\dagger} \\ \kappa^{*\dagger} & 0 \end{pmatrix}. \qquad \blacksquare$$

THEOREM 3.3. *Let $\phi: X \to X$ be a morphism of an additive category $\mathscr{C}$, and let $i \geq 0$ be an integer. If $\kappa: K \to X$ is a kernel and $\lambda: X \to L$ is a cokernel of $\phi^i: X \to X$, then ϕ has a Drazin inverse ϕ^D in $\mathscr{C}$ and $i \geq$ Drazin index of ϕ if and only if*

$$\mathscr{D} = \begin{pmatrix} \phi & \lambda \\ \kappa & 0 \end{pmatrix} : (X, K) \to (X, L)$$

is invertible in $\mathscr{M}_{\mathscr{C}}$. In this case, $\kappa\lambda: K \to L$ is invertible in $\mathscr{C}$, and

$$\mathscr{D}^{-1} = \begin{pmatrix} \phi^D & \lambda(\kappa\lambda)^{-1} \\ (\kappa\lambda)^{-1}\kappa & -(\kappa\lambda)^{-1}\kappa\phi\lambda(\kappa\lambda)^{-1} \end{pmatrix} : (X, L) \to (X, K).$$

Proof. Suppose that $\mathscr{D}$ is invertible with inverse

$$\mathscr{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (X, L) \to (X, K).$$

Then

$$\begin{pmatrix} \phi\alpha+\lambda\gamma & \phi\beta+\lambda\delta \\ \kappa\alpha & \kappa\beta \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_K \end{pmatrix} : (X,K) \to (X,K),$$

$$\begin{pmatrix} \alpha\phi+\beta\kappa & \alpha\lambda \\ \gamma\phi+\delta\kappa & \gamma\lambda \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_L \end{pmatrix} : (X,L) \to (X,L).$$

Since $\gamma\lambda = 1_L$ and $\phi\beta + \lambda\delta = 0$, then $\gamma\phi\beta + \delta = 0$ and $\delta = -\gamma\phi\beta$. Since $\alpha\phi + \beta\kappa = 1_X$ and $\kappa\phi^i = 0$, then $\alpha\phi^{i+1} = \phi^i$; similarly, since $\phi\alpha + \lambda\gamma = 1_X$ and $\phi^i\lambda = 0$, then $\phi^{i+1}\alpha = \phi^i$; also, $\kappa\alpha = 0$ and $\alpha\phi + \beta\kappa = 1_X$ requires that $\alpha\phi\alpha = \alpha$. Since $\gamma\phi + \delta\kappa = 0$, then $\gamma\phi^{i+1} = 0$, $\gamma\phi^i = \gamma\phi^{i+1}\alpha = 0$, and $\gamma = \xi\kappa$ for some $\xi: L \to K$. Similarly, $\beta = \lambda\eta$ for some $\eta: L \to K$. Thus, $1_L = \gamma\lambda = (\xi\kappa)\lambda = \xi(\kappa\lambda)$, $1_K = \kappa\beta = \kappa(\lambda\eta) = (\kappa\lambda)\eta$, and $\kappa\lambda$ is invertible with inverse $\xi = \eta$. That is, $\beta\kappa = \lambda\eta\kappa = \lambda\xi\kappa = \lambda\gamma$, $\alpha\phi + \beta\kappa = 1_X = \phi\alpha + \lambda\gamma$, and $\alpha\phi = \phi\alpha$. In summary, $\alpha\phi = \phi\alpha$, $\alpha\phi\alpha = \alpha$, and $\phi^i\alpha\phi = \phi^i$; that is, α is the Drazin inverse of ϕ and $i \geqslant$ Drazin index. Moreover, $\beta = \lambda(\kappa\lambda)^{-1}$, $\gamma = (\kappa\lambda)^{-1}\kappa$, and $\delta = -(\kappa\lambda)^{-1}\kappa\phi\lambda(\kappa\lambda)^{-1}$.

Conversely, suppose that ϕ has the Drazin inverse ϕ^D in $\mathscr{C}$ and that $i \geqslant$ Drazin index of ϕ. By Theorem 2, $\kappa\lambda: K \to L$ is invertible. In particular, the matrix $\mathscr{A}$ above is defined for $\alpha = \phi^D$, $\beta = \lambda(\kappa\lambda)^{-1}$, $\gamma = (\kappa\lambda)^{-1}\kappa$, and $\delta = -(\kappa\lambda)^{-1}\kappa\phi\lambda(\kappa\lambda)^{-1}$. Now, since $\phi\phi^D + \lambda(\kappa\lambda)^{-1}\kappa = 1_X$, $\kappa\phi^D = \kappa\phi^i(\phi^D)^{i+1} = 0$, $\kappa\lambda(\kappa\lambda)^{-1} = 1_K$, and

$$\begin{aligned} &\phi\lambda(\kappa\lambda)^{-1} - \lambda(\kappa\lambda)^{-1}\kappa\phi\lambda(\kappa\lambda)^{-1} \\ &= \left[1_X - \lambda(\kappa\lambda)^{-1}\kappa\right]\phi\lambda(\kappa\lambda)^{-1} \\ &= (\phi\phi^D)\phi\lambda(\kappa\lambda)^{-1} = \phi(\phi^D)^i(\phi^i\lambda)(\kappa\lambda)^{-1} = 0, \end{aligned}$$

we have $\mathscr{D}\mathscr{A} = 1_{(X,K)}$. Similarly, $\mathscr{A}\mathscr{D} = 1_{(X,L)}$. That is, $\mathscr{D}$ is invertible with inverse $\mathscr{A}$. ∎

Corollary 3. *Let $\phi: X \to X$ be a morphism of an additive category $\mathscr{C}$. If $\kappa: K \to X$ is a kernel and $\lambda: X \to L$ is a cokernel of ϕ, then ϕ has a group inverse in $\mathscr{C}$ if and only if*

$$\mathscr{G} = \begin{pmatrix} \phi & \lambda \\ \kappa & 0 \end{pmatrix} : (X,K) \to (X,L)$$

is invertible in $\mathscr{M}_{\mathscr{C}}$. In this case, $\kappa\lambda: K \to L$ is invertible in $\mathscr{C}$, and

$$\mathscr{G}^{-1} = \begin{pmatrix} \phi^{\#} & \lambda(\kappa\lambda)^{-1} \\ (\kappa\lambda)^{-1}\kappa & 0 \end{pmatrix} : (X, L) \to (X, K).$$

Proof. This corollary is the case $i = 1$ of Theorem 3.3. ■

4. EP MORPHISMS

Let $\phi: X \to X$ be a morphism in an additive category with involution $*$. If $\kappa: K \to X$ is a kernel of ϕ, we say that ϕ is EP provided that κ is also a kernel of ϕ^*. (See, for example, [30, p. 130], [2, p. 163], [4, p. 31], [5, p. 74], [12, p. 242], [15], [16], [17], [20, p. 584], [21], [22], [23, p. 674]; compare also [29].)

FACT. *If ϕ is* EP, *then $\kappa^*: X \to K$ is a cokernel of ϕ.*

Proof. First, $\phi\kappa^* = (\kappa\phi^*)^* = 0^* = 0$. Second, if $\phi\eta = 0$, then $\eta^*\phi^* = 0$, $\eta^* = \xi\kappa$, and $\eta = \kappa^*\xi^*$. Finally, if $\kappa^*\mu = 0$, then $\mu^*\kappa = 0$ and, since κ is monic, $\mu^* = 0$, $\mu = 0$.

THEOREM 4. *Let $\phi: X \to X$ be a morphism of an additive category with involution $*$, and suppose that $\kappa: K \to X$ is a kernel of ϕ. If ϕ is* EP, *then $\phi^{\#}$ exists if and only if $\phi^{\dagger}$ exists, and in this case $\phi^{\#} = \phi^{\dagger}$. If $\phi^{\#}$ and $\phi^{\dagger}$ exist and $\phi^{\#} = \phi^{\dagger}$, then ϕ is* EP.

Proof. Let $\phi: X \to X$ be EP. By the above Fact, $\kappa^*: X \to X$ is a cokernel of ϕ. By Corollary 3, $\phi^{\#}$ exists iff

$$\mathscr{G} = \begin{pmatrix} \phi & \kappa^* \\ \kappa & 0 \end{pmatrix} \text{ is invertible}$$

iff

$$\Phi = \begin{pmatrix} \phi & \kappa^* \\ (\kappa^*)^* & 0 \end{pmatrix} \text{ is invertible}$$

iff $\phi^{\dagger}$ exists. In this case, $\mathscr{G}^{-1} = \Phi^{-1}$ and $\phi^{\#} = \phi^{\dagger}$.

On the other hand, let $\phi^{\#}$ and $\phi^{\dagger}$ both exist and let $\phi^{\#} = \phi^{\dagger}$. In particular, since $\phi^{\#}$ exists, then ϕ has a cokernel $\lambda : L \to X$ and

$$\phi\left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1} = \phi^{\#} = \phi^{\dagger} = \phi^*(\phi\phi^* + \kappa^*\kappa)^{-1}.$$

It follows that κ is also a kernel of ϕ^*, and therefore that ϕ is EP. ■

EXAMPLE. Let $\mathscr{M}_{\mathbb{Z}/2}$ be the category of finite matrices over $\mathbb{Z}/2\mathbb{Z}$ with the involution * of matrix transpose. The object 3 in this category provides choices that illustrate each of the six possibilities allowed by Theorem 4:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies $\phi^{\#} = \phi^{\dagger} = \phi$, and ϕ is EP.

$$\phi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies neither $\phi^{\#}$ nor $\phi^{\dagger}$ exists, but ϕ is EP.

$$\phi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies $\phi^{\#} = \phi$, but $\phi^{\dagger}$ does not exist; ϕ is not EP.

$$\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies $\phi^{\dagger} = \phi^*$, but $\phi^{\#}$ does not exist; ϕ is not EP.

$$\phi = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies $\phi^{\#} = \phi \neq \phi^* = \phi^{\dagger}$, and ϕ is not EP.

$$\phi = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies neither $\phi^{\#}$ nor $\phi^{\dagger}$ exists; ϕ is not EP.

We conclude by extending to categories a familiar result about EP matrices. (See, for example, [23, p. 677].)

COROLLARY 4. *Let the conditions be as in Theorem 4. If ϕ is EP and $\phi^{\dagger}$ exists, then $\phi^{\dagger}\beta = \beta\phi^{\dagger}$ whenever $\phi\beta = \beta\phi$.*

Proof. By hypothesis and Theorems 2 and 4, $\phi^{\#}$ exists, $\phi^{\#} = \phi^{\dagger}$, and ϕ has a cokernel $\lambda : X \to L$. Thus, if $\phi\beta = \beta\phi$, then

$$\left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]\phi\beta = \phi^3\beta = \beta\phi^3 = \beta\phi\left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right],$$

and

$$\begin{aligned}\phi^{\dagger}\beta = \phi^{\#}\beta &= \left\{\left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi\right\}\beta \\ &= \left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\phi\beta = \left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1}\beta\phi \\ &= \phi\beta\left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1} = \beta\phi\left[\phi^2 + \lambda(\kappa\lambda)^{-1}\kappa\right]^{-1} \\ &= \beta\phi^{\#} = \beta\phi^{\dagger}.\end{aligned}$$

■

REFERENCES

1 A. Ben-Israel, A Cramer rule for least-norm solutions of consistent linear equations, *Linear Algebra Appl.* 40:223–226 (1982).

2 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, New York, 1974.

3 J. W. Blattner, Bordered matrices, *J. Soc. Indust. Appl. Math.* 10:528–536 (1962).

4 T. L. Boullion and P. L. Odell, *Generalized Inverse Matrices*, Wiley, New York, 1971.

5 S. L. Campbell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.

6 M. P. Drazin, Pseudo inverses in associative rings and semigroups, *Amer. Math. Monthly* 65:506–514 (1958).

7 M. J. Englefield, The commuting inverses of a square matrix, *Proc. Cambridge Philos. Soc.* 62:667–671 (1966).

8 I. Erdelyi, On the matrix equation $Ax = \lambda Bx$, *J. Math. Anal. Appl.* 17:119–132 (1967).

9 M. Fiedler and T. L. Markham, Completing a matrix when certain entries of its inverse are specified, *Linear Algebra Appl.* 74:225–237 (1986).

10 B. Germain-Bonne, Calcul de pseudo-inverses, *Rev. Français Informat. Recherche Opér.*, 1969, pp. 3–14.

11 A. J. Goldman and M. Zelen, Weak generalized inverses and minimum variance linear unbiased estimation, *J. Res. Nat. Bur. Standards Sect. B* 68B:151–172 (1964).

12 R. E. Hartwig, Block generalized inverses, *Arch. Rational Mech. Anal.* 61:197–251 (1976).

13 J. Z. Hearon, Construction of EPr generalized inverses by inversion of nonsingular matrices, *J. Res. Nat. Bur. Standards Sect. B* 71B:57–60 (1967).

14 J. Z. Hearon, On the singularity of a certain bordered matrix, *SIAM J. Appl. Math* 15:1413–1421 (1967).

15 I. J. Katz and M. H. Pearl, On EPr and normal EPr matrices, *J. Res. Nat. Bur. Standards Sect. B* 70B:47–77 (1966).

16 Ar. Meenakshi, On EPr matrices with entries from an arbitrary field, *Linear and Multilinear Algebra* 9:159–164 (1980).

17 C. D. Meyer, Jr., Some remarks on EPr matrices and generalized inverses, *Linear Algebra Appl.* 3:275–278 (1970).

18 L. Mirsky, *An Introduction to Linear Algebra*, Clarendon, Oxford, 1955.

19 E. H. Moore, On the reciprocal of the general algebraic matrix (Abstract), *Bull. Amer. Math. Soc.* 26:394–395.

20 M. H. Pearl, Generalized inverses of matrices with entries taken from an arbitrary field, *Linear Algebra Appl.* 1:571–587 (1968).

21 M. H. Pearl, On normal and EPr matrices, *Michigan Math. J.* 6:1–5 (1959).

22 M. H. Pearl, On normal EPr matrices, *Michigan Math. J.* 8:33–37 (1961).

23 M. H. Pearl, On Generalized inverse matrices, *Proc. Cambridge Philos. Soc.* 62:673–677 (1966).

24 R. Penrose, A generalized inverse for matrices, *Proc. Cambridge Philos. Soc.* 51:406–413 (1955).

25 R. Puystjens and D. W. Robinson, The Moore-Penrose inverse of a morphism in an additive category, *Comm. Algebra* 12(3):287–299 (1984).

26 R. Puystjens and D. W. Robinson, The Moore-Penrose inverse of a morphism with factorization, *Linear Algebra Appl.* 40:129–141 (1981).

27 W. T. Reid, Principal solutions of non-oscillatory linear differential systems, *J. Math. Anal. Appl.* 9:397–423 (1964).

28 P. Robert, On the group-inverse of a linear transformation, *J. Math. Anal. Appl.* 22:658–669 (1968).

29 D. W. Robinson and R. Puystjens, EP morphisms, *Linear Algebra Appl.* 64:157–174 (1985).

30 H. Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, Noordhoff, Groningen, 1950.

31 G. C. Verghese, A "Cramer rule" for the least-norm, least-squared-error solution of inconsistent linear equations, *Linear Algebra Appl.* 48:315–316 (1982).

Received 15 August 1986; revised 3 December 1986